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Problems in the derivations of the renormalization group equation for the low momentum nucleon interactions

Koji Harada*

Department of Physics, Kyushu University

Fukuoka 810-8560 Japan

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Abstract

We carefully examine all the four derivations of the renormalization group equation (RGE) for the so-called $V_{\mathrm{low\ k}}$ potential, given by Bogner, *et. al.*[nucl-th/0111042]. Two derivations based on the "semi-group composition law" are shown to be unjustified, while the other two based on the completeness relation of the model space must be modified if there are bound states. It is however shown that the RGE is unchanged if the bound state wavefunctions in the reduced theory are required to have the same low-momentum components as those in the original theory. Several aspects of the $V_{\mathrm{low\ k}}$ approach are also discussed.

*Electronic address: harada@phys.kyushu-u.ac.jp

I. INTRODUCTION

Recently there has been great interest in studying nucleon-nucleon (NN) interactions by using Effective Field Theory (EFT) [1, 2]. EFT is based on the very simple idea that the effects of short-distance physics can be simulated by local (contact) interactions, so that a typical (nuclear) EFT-based potential consists of a series of local interactions (delta function potentials and their derivatives) and a long-distance part (Yukawa potential due to one-pion exchange). EFT is valid only below a certain scale, the cutoff, and the coupling constants of the local interactions are determined as functions of the cutoff so that the physical observables are independent of the cutoff. Lowering the cutoff amounts to the "integrating" the short-distance fluctuations, leading to the change of the coupling constants of the local interactions. This is nothing but Wilsonian renormalization group (RG) idea [3]. EFT together with the RG idea is a promising alternative to the conventional approach based on phenomenological NN potentials. It is a model independent, systematically improvable approach, and is related to QCD through chiral symmetry. See Refs. [4, 5, 6, 7] for reviews.

One might however think that the use of the "realistic" potentials [8, 9, 10, 11], which describe thousands of data with $E_{lab} \lesssim 350$ MeV very accurately, together with the RG idea, would be *more* effective, though such an approach is inevitably phenomenological. The so-called $V_{\rm low\ k}$ approach [12, 13, 14, 15, 16, 17] (in its original form) exactly goes along this strategy.

The $V_{\rm low\ k}$ approach originally emerged as an application of the model space reduction methods developed in shell-model theory [12]. The model space reduction methods apparently resemble RG transformations; the effects of the higher-energy states are taken into account as effective interactions. It has been shown that "realistic" potentials evolve to a universal low-energy potential by the model space reduction [13]. It appears to conform to the general idea of "universality" in the RG theory. On the other hand, the universal potential may be useful also in a practical sense, because it is much smoother than the "realistic" potentials so that one may avoid calculating the Brueckner G-matrix [18, 19, 20, 21], which is energy and nucleus dependent.

A relatively simple RG equation (RGE), which is claimed to be satisfied by the $V_{\rm low\ k}$, has been known;

$$\frac{d}{d\Lambda} V^{\Lambda}_{\rm low\ k}(k',k) = \frac{2}{\pi} \frac{V^{\Lambda}_{\rm low\ k}(k',\Lambda) T^{\Lambda}(\Lambda,k;\Lambda^2)}{1-(k/\Lambda)^2}, \tag{1.1}$$

where Λ is the (floating) cutoff, k and k' are the relative momentum of incoming and outgoing nucleons respectively, and T is the T-matrix. The superscript Λ indicates that the quantity is cutoff dependent. Note that we put the superscript Λ on the left-on-shell T-matrix to indicate explicitly that it is, unlike the right-on-shell T-matrix, a Λ-dependent quantity. It is important to note that "the RG equation lies at the heart of the approach [13]." There are several model space techniques and, according to Refs. [12, 14, 15, 16, 22], they all satisfy the same RG equation. In this sense, it is considered that they are all equivalent and the universal low-energy potential is unique. Because this is a very nontrivial assertion, it is important to examine the derivations of the RG equation carefully.

To our best knowledge, however, the derivations of the RG equations presented in Ref. [14] do not seem to have been scrutinized, though there are many applications. In this paper, we show that each of the four derivations presented in Ref. [14] contains a serious defect. Our arguments are not technically intricate nor too mathematical so that those who are familiar with the derivations can easily follow them.

We do not claim that their RGE (1.1) is wrong; we only claim that all the existent *derivations* are either unjustified or, if there are bound states, insufficient. It is however shown that extra terms arising from the existence of the bound states cancel, keeping Eq. (1.1) unmodified [23], if the bound state wavefunctions in the reduced theory are required to have the same low-momentum components as those in the original theory.

We notice that most of the numerical results given in the literature are not derived directly from the RGE, but from one of the model-space reduction methods such as Lee-Suzuki (LS) method [24] or Kuo-Lee-Ratcliff (KLR) method [25]. We emphasize that numerical results obtained with these methods are not affected by the present work. The procedure is independent of the actual form of the RGE, and the effective interactions obtained by these methods in fact satisfy the condition mentioned above, and hence Eq.(1.1).

We assume that the reader is familiar with Ref. [14], though we recapitulate the main points. We follow their notations as closely as possible. In several places, we need to supplement their arguments by providing the missing intermediate steps which we think they actually took.

We start with the definition of the so-called $\hat{Q}$-box and the KLR method in Sec. II. By supplementing the intermediate steps, we point out explicitly where they took a wrong step, invalidating their second and third derivations. In Sec. III, we show that their completeness

relation in the "model space" is wrong when bound states are present, invalidating the first derivation. It is however shown that, if the bound state wavefunctions in the reduced theory have the same low-momentum components as those in the original theory, the extra terms cancel each other, and the RGE is not affected by the presence of bound states. The fourth derivation is also based on the same completeness relation, so that it is also invalid in the presence of bound states as we show in Sec. IV. Again, the final result, Eq. (1.1), can be derived without change under the same condition as in Sec. III. In Sec. V, we summarize our results and discuss several related issues.

II. $\hat{Q}$-BOX AND THE KUO-LEE-RATCLIFF METHOD

Let us start with the "bare" Lippmann-Schwinger equation for the (off-shell) T-matrix,

$$T = V_{NN} + V_{NN} G_0 T, \tag{2.1}$$

where V_{NN} is a "realistic" potential. In the partial-wave notation, this may be written as

$$T(k', k; \omega) = V_{NN}(k', k) + \frac{2}{\pi} \mathcal{P} \int_0^\infty V_{NN}(k, p) \frac{p^2 dp}{\omega - p^2} T(p, k; \omega), \tag{2.2}$$

where, following Ref. [14], we consider the standing wave boundary condition, but the following arguments are largely independent of it. (In this paper, we use units with $\hbar = c = M_N = 1$.) Physically speaking, the upper limit should be considered as a very large cutoff Λ_0 rather than the infinity.

One would like to replace the "bare" Lippmann-Schwinger equation by a cutoff one, which describes the same physics at low energies. For this purpose, let us introduce the projection operators, P and Q, satisfying $P + Q = 1$, $PQ = QP = 0$, $P^2 = P$, and $Q^2 = Q$. The operator P, which projects states to the low-momentum space, may be written as

$$P = \frac{2}{\pi} \int_0^\Lambda |k\rangle k^2 dk \langle k|, \tag{2.3}$$

where $|k\rangle$ stands for the plane wave in the partial-wave notation with the normalization,

$$\langle k'|k\rangle = \frac{\pi}{2} \frac{\delta(k' - k)}{k' k}. \tag{2.4}$$

The so-called $\hat{Q}$-box may be defined as

$$\hat{Q}^\Lambda(k', k; \omega) = V_{NN}(k', k) + \frac{2}{\pi} \mathcal{P} \int_\Lambda^\infty V_{NN}(k, p) \frac{p^2 dp}{\omega - p^2} \hat{Q}^\Lambda(p, k; \omega), \tag{2.5}$$

which is the part of the T-matrix, all of the intermediate states of which are in the Q-space. We put the superscript Λ to indicate that it depends on the cutoff. By using $\hat{Q}$-box, the T-matrix may be written as

$$T(k',k;\omega) = \hat{Q}^{\Lambda}(k',k;\omega) + \frac{2}{\pi}\mathcal{P}\int_0^{\Lambda} \hat{Q}^{\Lambda}(k',p;\omega)\frac{p^2dp}{\omega - p^2}T(p,k;\omega). \tag{2.6}$$

Note that $\hat{Q}$-box is energy dependent. If we restrict $k',\ k \le \Lambda$, then $\hat{Q}$-box may be identified with the Wilsonian effective potential V^{Λ}_{WRG} satisfying

$$\frac{d}{d\Lambda}V^{\Lambda}_{WRG}(k',k;\omega) = \frac{2}{\pi}\frac{V^{\Lambda}_{WRG}(k',\Lambda;\omega)V^{\Lambda}_{WRG}(\Lambda,k;\omega)}{1-\omega/\Lambda^2}, \tag{2.7}$$

considered by Birse *et. al.* [26]. See Refs. [27, 28, 29] for the field theoretical formulation. See also Ref. [30] for the comparison of V_{WRG} with $V_{\text{low k}}$.

First of all, it is important to note that for the *fully off-shell* T-matrix there is *no* such energy-independent potential V^{Λ} that satisfies

$$T(k',k;\omega) = V^{\Lambda}(k',k) + \frac{2}{\pi}\mathcal{P}\int_0^{\Lambda} V^{\Lambda}(k',p)\frac{p^2dp}{\omega - p^2}T(p,k;\omega). \tag{2.8}$$

Actually, the Λ-independence of the (off-shell) T-matrix leads to the RGE (2.7), which cannot be satisfied by an energy-independent potential.

On the other hand, for the *half-on-shell* T-matrix, there exists an energy-independent potential $V_{\text{low k}}$,

$$T(k',k;k^2) = V^{\Lambda}_{\text{low k}}(k',k) + \frac{2}{\pi}\mathcal{P}\int_0^{\Lambda} V^{\Lambda}_{\text{low k}}(k',p)\frac{p^2dp}{k^2 - p^2}T(p,k;k^2). \tag{2.9}$$

This may serve as the definition of $V_{\text{low k}}$.

A concrete procedure of constructing such a potential is given by the KLR folded diagram theory [25]. Starting with Eq. (2.6) with $\omega = k^2$, we have

$$T(k',k;k^2) = \hat{Q}^{\Lambda}(k',k;k^2) + \frac{2}{\pi}\mathcal{P}\int_0^{\Lambda} \hat{Q}^{\Lambda}(k',p;k^2)\frac{p^2dp}{k^2 - p^2}T(p,k;k^2), \tag{2.10}$$

which should be compared with Eq. (2.9). Although $\hat{Q}^{\Lambda}(k',k;k^2)$ almost satisfies Eq. (2.9) for $V^{\Lambda}_{\text{low k}}$, $\hat{Q}^{\Lambda}(k',p;k^2)$ in the integrand depends on three variables. We therefore write

$$V^{\Lambda}_{\text{low k}}(k',k) = \hat{Q}^{\Lambda}(k',k;k^2) + \sum_{i=2}^{\infty}\Delta V^{(i)}(k',k), \tag{2.11}$$

where $\Delta V^{(i)}$ is the correction term coming from the folded diagrams and is of order of the i-th power of $\hat{Q}$.

Substituting it into Eq. (2.9) and comparing with Eq. (2.6), we have

$$\sum_{i=2}^{\infty}\Delta V^{(i)}(k',k)=\int_p\frac{\Delta Q^\Lambda_{(k)}(k',p)}{k^2-p^2}\hat{Q}^\Lambda(p,k)+\int_p\int_{p'}\frac{\Delta Q^\Lambda_{(k)}(k',p')\Delta Q^\Lambda_{(k)}(p',p)}{(k^2-p^2)(k^2-p'^2)}T(p,k;k^2)$$
$$+\int_p\int_{p'}\frac{\Delta Q^\Lambda_{(k)}(k',p')\hat{Q}^\Lambda(p',p;p^2)}{(k^2-p^2)(k^2-p'^2)}T(p,k;k^2)-\int_p\sum_{i=2}^{\infty}\Delta V^{(i)}(k',p)\frac{T(p,k;k^2)}{k^2-p^2},\tag{2.12}$$

where the notations $\int_p\equiv\frac{2}{\pi}\mathcal{P}\int_0^\Lambda p^2dp$ and

$$\Delta Q^\Lambda_{(k)}(k',p)\equiv\hat{Q}^\Lambda(k',p;k^2)-\hat{Q}^\Lambda(k',p;p^2)\tag{2.13}$$

are introduced. Eq. (2.12) determines $\Delta V^{(i)}$ iteratively.

Having shown the existence of the $V^\Lambda_{\text{low k}}$ which satisfies Eq. (2.9), we now turn to the question about what RGE it satisfies. The authors of Ref. [14] emphasized the "semi-group composition law," and used the following relation,

$$\hat{Q}^{\Lambda-\delta\Lambda}(k',k;p^2)=V^\Lambda_{\text{low k}}(k',k)-\delta\Lambda\frac{2}{\pi}\frac{V^\Lambda_{\text{low k}}(k',\Lambda)V^\Lambda_{\text{low k}}(\Lambda,k)}{1-(p/\Lambda)^2}+\mathcal{O}(\delta\Lambda^2).\tag{2.14}$$

This infinitesimal form is derived from the following finite form,

$$\hat{Q}^{\bar\Lambda}(k',k,p^2)=V^\Lambda_{\text{low k}}(k',k)+\frac{2}{\pi}\mathcal{P}\int_{\bar\Lambda}^\Lambda V^\Lambda_{\text{low k}}(k',\bar{k})\frac{\bar{k}^2d\bar{k}}{p^2-\bar{k}^2}\hat{Q}^{\bar\Lambda}(\bar{k},k,p^2),\tag{2.15}$$

where $\bar\Lambda\le\Lambda$ is another scale. One might think that this is the result of integrating Eq. (2.5) up to a certain scale and replacing the "bare" potential $V_{NN}(k',k)$ with the "effective" one, $V^\Lambda_{\text{low k}}(k',k)$, accordingly. Eq. (2.14) can be easily derived by putting $\bar\Lambda=\Lambda-\delta\Lambda$.

It is however easy to see that *there is no such* $V^\Lambda_{\text{low k}}$ *that satisfies* Eq. (2.15) *for an arbitrary* $\Lambda<\Lambda_0$. Consider the limit $\bar\Lambda\to0$. From Eq. (2.5) we see

$$\lim_{\bar\Lambda\to0}\hat{Q}^{\bar\Lambda}(k',k;p^2)=T(k',k;p^2),\tag{2.16}$$

so that Eq. (2.15) becomes Eq. (2.8) with $V^\Lambda(k',k)$ being replaced by $V_{\text{low k}}(k',k)$, but, as we emphasized, there is no such energy-independent potential that satisfies Eq. (2.8). We have thus shown that Eq. (2.14) cannot be valid. This is the defect in the second derivation in Ref. [14]. The same Eq. (2.14) is used in the third derivation too in a crucial way, so that the third one has the same defect.

III. COMPLETENESS IN THE MODEL SPACE

Since $V_{\text{low k}}$ preserves the half-on-shell T-matrix, one may think of another way of deriving the RGE, namely, from the Λ-independence of the half-on-shell T-matrix in Eq. (2.9). In the case of V_{WRG}, the RGE can be easily derived from the Λ-independence of the fully off-shell T-matrix. On the other hand, one cannot follow the similar manipulation for $V_{\text{low k}}$, and the authors of Ref. [14] invoked information on the cutoff state vectors. But, as we will show, the relation they used is wrong when bound states are present.

Consider the "bare" Schrödinger equation,

$$(H_0 + V_{NN})|\Psi_k\rangle = k^2|\Psi_k\rangle, \tag{3.1}$$

for a scattering state with energy $0 \le k^2 < \Lambda^2$, and the corresponding reduced one,

$$(H_0 + V^{\Lambda}_{\text{low k}})|\chi^{\Lambda}_k\rangle = k^2|\chi^{\Lambda}_k\rangle. \tag{3.2}$$

The potential $V^{\Lambda}_{\text{low k}}$ is a P-to-P operator, which may be constructed by the KLR folded diagram theory or the LS similarity transformation. In the both cases, the state $|\chi^{\Lambda}_k\rangle$ is related to the original state by the projection,

$$|\chi^{\Lambda}_k\rangle = P|\Psi_k\rangle. \tag{3.3}$$

This condition is nothing but the Λ-independence of the half-on-shell T-matrix in the reduced theory. Actually, by applying $\langle k'|$ with $k' < \Lambda$ to Eqs. (3.1) and (3.2), we get

$$\langle k'|V_{NN}|\Psi_k\rangle = (k^2 - k'^2)\langle k'|\Psi_k\rangle \quad \text{and} \quad \langle k'|V_{\text{low k}}|\chi^{\Lambda}_k\rangle = (k^2 - k'^2)\langle k'|\chi^{\Lambda}_k\rangle, \tag{3.4}$$

but the left hand sides of these equations are by assumption equal to the half-on-shell T-matrix $T(k', k; k^2)$. Therefore the right hand sides are also equal, and taking into account that k' is arbitrary, we see that Eq. (3.3) is valid. On the other hand, the preservation of the half-on-shell T-matrix follows Eq. (3.3).

Their first derivation goes as follows: From the cutoff Schrödinger equation (3.2), one gets the corresponding Lippmann-Schwinger equation,

$$|\chi^{\Lambda}_k\rangle = |k\rangle + \frac{2}{\pi}\mathcal{P}\int_0^{\Lambda} dp|p\rangle\frac{p^2}{k^2 - p^2}T(p, k; k^2), \tag{3.5}$$

where k is assumed to be small, $k < \Lambda$. Note that the upper limit of the integral is Λ to be consistent with Eq.(3.3). Requiring that the half-on-shell T-matrix is Λ-independent,

$$\frac{d}{d\Lambda}T(k',k;k^2) = \frac{d}{d\Lambda}\langle k'|V^{\Lambda}_{\text{low k}}|\chi^{\Lambda}_k\rangle = 0, \tag{3.6}$$

for $k' < \Lambda$ and using

$$\frac{d}{d\Lambda}|\chi^{\Lambda}_k\rangle = \frac{2}{\pi}|\Lambda\rangle\frac{\Lambda^2}{k^2-\Lambda^2}T(\Lambda,k;k^2), \tag{3.7}$$

one gets

$$0 = \langle k'|\frac{dV^{\Lambda}_{\text{low k}}}{d\Lambda}|\chi^{\Lambda}_k\rangle + \frac{2}{\pi}\langle k'|V^{\Lambda}_{\text{low k}}|\Lambda\rangle\frac{\Lambda^2}{k^2-\Lambda^2}T(\Lambda,k;k^2). \tag{3.8}$$

The last term may be rewritten by using

$$\frac{\Lambda^2}{k^2-\Lambda^2}T(\Lambda,k;k^2) = \langle\Lambda|V^{\Lambda}_{\text{low k}}\frac{\Lambda^2}{H_0+V^{\Lambda}_{\text{low k}}-\Lambda^2}|\chi^{\Lambda}_k\rangle. \tag{3.9}$$

Let us introduce the operator,

$$J \equiv \frac{2}{\pi}\int_0^{\Lambda}|\chi^{\Lambda}_k\rangle k^2 dk\langle\tilde{\chi}^{\Lambda}_k|, \tag{3.10}$$

where we introduced a bi-orthogonal basis $\langle\tilde{\chi}^{\Lambda}_k|$ for each state $|\chi^{\Lambda}_k\rangle$,

$$\langle\tilde{\chi}^{\Lambda}_{k'}|\chi^{\Lambda}_k\rangle = \frac{\pi}{2}\frac{\delta(k'-k)}{k'k}, \tag{3.11}$$

because $V^{\Lambda}_{\text{low k}}$ is not Hermitian. Note that J is a projection operator, $J^2 = J$. One thus obtains

$$\langle k'|\frac{dV^{\Lambda}_{\text{low k}}}{d\Lambda}J|k\rangle = -\frac{2}{\pi}\langle k'|V^{\Lambda}_{\text{low k}}|\Lambda\rangle\langle\Lambda|V^{\Lambda}_{\text{low k}}\frac{\Lambda^2}{H_0+V^{\Lambda}_{\text{low k}}-\Lambda^2}J|k\rangle. \tag{3.12}$$

If J were the identity operator in the P-space, as the authors of Ref. [14] claimed, then the identity ,

$$\langle\Lambda|V^{\Lambda}_{\text{low k}}\frac{\Lambda^2}{H_0+V^{\Lambda}_{\text{low k}}-\Lambda^2}|k\rangle = T^{\Lambda}(\Lambda,k;\Lambda^2)\frac{\Lambda^2}{k^2-\Lambda^2}, \tag{3.13}$$

would lead us to

$$\frac{d}{d\Lambda}V^{\Lambda}_{\text{low k}}(k',k) = -\frac{2}{\pi}V^{\Lambda}_{\text{low k}}(k',\Lambda)T^{\Lambda}(\Lambda,k;\Lambda^2)\frac{\Lambda^2}{k^2-\Lambda^2}, \tag{3.14}$$

which is nothing but Eq. (1.1).

In the presence of bound states, however, the operator J cannot be the identity operator,

$$J + \sum_i \left|\chi^{\Lambda}_{B_i}\rangle\langle\tilde{\chi}^{\Lambda}_{B_i}\right| = P, \tag{3.15}$$

where $|\chi^{\Lambda}_{B_i}\rangle$ are state vectors for the bound states in the reduced theory,

$$(H_0 + V^{\Lambda}_{\text{low k}})|\chi^{\Lambda}_{B_i}\rangle = -k^2_{B_i}|\chi^{\Lambda}_{B_i}\rangle, \tag{3.16}$$

and $\langle\tilde{\chi}^{\Lambda}_{B_i}|$ is its conjugate bi-orthogonal state,

$$\langle\tilde{\chi}^{\Lambda}_{B_i}|\chi^{\Lambda}_{B_j}\rangle = \delta_{ij}. \tag{3.17}$$

Since the first derivation in Ref. [14] uses the claim that J is the identity operator in the P-space in an essential way, it contains a serious defect in the presence of bound states.

The point is that completeness of the eigenstates of an operator concerns the whole spectrum, including the bound states.

In the original applications of the KLR folded diagram theory or LS similarity transformation theory to the shell model problems, the spectrum of the Hamiltonian is discrete and the model space is finite dimensional. Everything looks trivial about the completeness. In the scattering problem, however, even though one is interested in continuous states, bound states must be included if one talks about completeness.

One might claim that the reduced theory is designed to reproduce the scattering amplitudes, so that it does not need to include the bound states in the spectrum. It is however not correct because the scattering amplitudes in general reflect some information about the bound states. A good example is provided by the well known Levinson's theorem[31], which states that the low-energy scattering data (the phase shift at zero momentum) for a well-behaved potential knows the number of the bound states. If the reduced potential $V_{\text{low k}}$ reproduces the half-on-shell, hence the on-shell T-matrix, it must support the bound states because the existence of which is encoded in the scattering phase shift.

If one takes into account the correct completeness relation, Eq. (3.15), one finds the modified RGE given by

$$\begin{aligned}\frac{d}{d\Lambda}V^{\Lambda}_{\text{low k}}(k',k) \;=\; & \frac{2}{\pi}\frac{V^{\Lambda}_{\text{low k}}(k',\Lambda)T^{\Lambda}(\Lambda,k;\Lambda^2)}{1-(k/\Lambda)^2} \\ & + \sum_i \int_l \left(\frac{d}{d\Lambda}V^{\Lambda}_{\text{low k}}(k',l) - \frac{2}{\pi}\frac{V^{\Lambda}_{\text{low k}}(k',\Lambda)V^{\Lambda}_{\text{low k}}(\Lambda,l)}{1+(k_{B_i}/\Lambda)^2}\right)\chi^{\Lambda}_{B_i}(l)\left(\tilde{\chi}^{\Lambda}_{B_i}(k)\right)^*,\end{aligned} \tag{3.18}$$

where $\chi^{\Lambda}_{B_i}(k) = \langle k|\chi^{\Lambda}_{B_i}\rangle$ is the wave function in momentum space for the bound state, and $\tilde{\chi}^{\Lambda}_{B_i}(k)$ is its bi-orthogonal conjugate.

Interestingly, however, the extra terms are shown to cancel each other [42], if we assume that the state $|\chi^{\Lambda}_{B_i}\rangle$ is related to the state vector $|\Psi_{B_i}\rangle$ in the original theory,

$$(H_0 + V_{NN})|\Psi_{B_i}\rangle = -k^2_{B_i}|\Psi_{B_i}\rangle, \tag{3.19}$$

through the projection,

$$|\chi^{\Lambda}_{B_i}\rangle = P|\Psi_{B_i}\rangle, \tag{3.20}$$

in conjunction with Eq. (3.3). Note that the conditions Eqs. (3.20) and (3.3) are independent.

To see the cancellation, let us first note that the condition

$$\frac{d}{d\Lambda}\langle k'|V^{\Lambda}_{\text{low k}}|\chi^{\Lambda}_{B_i}\rangle = 0, \quad (0 \le k' < \Lambda), \tag{3.21}$$

and Eq. (3.20) are equivalent, as Eqs. (3.6) and (3.3) are. Then, from the bound-state Lippmann-Schwinger equation,

$$|\chi^{\Lambda}_{B_i}\rangle = \int_p |p\rangle \frac{-1}{p^2 + k^2_{B_i}}\langle p|V^{\Lambda}_{\text{low k}}|\chi^{\Lambda}_{B_i}\rangle, \tag{3.22}$$

we have

$$\frac{d}{d\Lambda}|\chi^{\Lambda}_{B_i}\rangle = -\frac{2}{\pi}|\Lambda\rangle \frac{\Lambda^2}{\Lambda^2 + k^2_{B_i}}\langle \Lambda|V^{\Lambda}_{\text{low k}}|\chi^{\Lambda}_{B_i}\rangle, \tag{3.23}$$

which corresponds to Eq. (3.7). From these equations, we have

$$\begin{aligned} 0 &= \langle k'|\frac{dV^{\Lambda}_{\text{low k}}}{d\Lambda}|\chi^{\Lambda}_{B_i}\rangle - \frac{2}{\pi}\frac{\langle k'|V^{\Lambda}_{\text{low k}}\rangle\langle \Lambda|V^{\Lambda}_{\text{low k}}|\chi^{\Lambda}_{B_i}\rangle}{1 + (k_{B_i}/\Lambda)^2} \\ &= \frac{2}{\pi}\int_l \left(\frac{d}{d\Lambda}V^{\Lambda}_{\text{low k}}(k', l) - \frac{2}{\pi}\frac{V^{\Lambda}_{\text{low k}}(k', \Lambda)V^{\Lambda}_{\text{low k}}(\Lambda, l)}{1 + (k_{B_i}/\Lambda)^2} \right) \chi^{\Lambda}_{B_i}(l), \end{aligned} \tag{3.24}$$

thus, the extra terms in Eq. (3.18) vanish identically.

In conclusion, though the first derivation in Ref. [14] does not take into account the existence of bound states at all, and the completeness relation is wrong when there are bound states, the RGE (1.1) is shown to be unmodified irrespective of the existence of bound states, if we assume Eq. (3.20).

IV. SPECTRAL REPRESENTATION FOR THE T MATRIX

The fourth derivation of the RGE is given in the appendix of Ref. [14]. The derivation is based on the following spectral representation for the T-matrix,

$$V^{\Lambda}_{\text{low k}}(k', k) = T(k', k; k^2) + \int_p T(k', p; p^2)\frac{1}{p^2 - k^2}T^{\Lambda}(p, k; p^2). \tag{4.1}$$

In the following, I will show how this representation is modified in the presence of bound states. Let us start with Eq. (2.9) in a bit different representation,

$$T(k',k;k^2) = V^\Lambda_{\text{low k}}(k',k) + \langle k'|V^\Lambda_{\text{low k}} G_0^{(P)}(k^2) T(k^2)|k\rangle, \tag{4.2}$$

where $G_0^{(P)}(k^2)$ is the free Green function in the P-space,

$$G_0^{(P)}(k^2) = \frac{P}{k^2 - H_0}. \tag{4.3}$$

Eq. (4.2) may be rewritten by using the full Green function in the P-space,

$$G^{(P)}(k^2) = \frac{P}{k^2 - H^\Lambda}, \quad H^\Lambda = H_0 + V^\Lambda_{\text{low k}}, \tag{4.4}$$

as

$$T(k',k;k^2) = V^\Lambda_{\text{low k}}(k',k) + \langle k'|V^\Lambda_{\text{low k}} G^{(P)}(k^2) V^\Lambda_{\text{low k}}|k\rangle. \tag{4.5}$$

If the operator J in Eq. (3.10) were the identity operator in the P-space, then, inserting J and using $H^\Lambda|\chi^\Lambda_p\rangle = p^2|\chi^\Lambda_p\rangle$, one would get

$$T(k',k;k^2) = V^\Lambda_{\text{low k}}(k',k) + \int_p \langle k'|V^\Lambda_{\text{low k}}|\chi^\Lambda_p\rangle \frac{1}{k^2 - p^2} \langle \tilde{\chi}^\Lambda_p|V^\Lambda_{\text{low k}}|k\rangle, \tag{4.6}$$

hence Eq. (4.1), with the identification $\langle k'|V^\Lambda_{\text{low k}}|\chi^\Lambda_p\rangle = T(k',p;p^2)$ and $\langle \tilde{\chi}^\Lambda_p|V^\Lambda_{\text{low k}}|k\rangle = T^\Lambda(p,k;p^2)$. As we have shown, however, the operator J is actually not the identity operator in the P-space in the presence of bound states, this manipulation is not justified. In the presence of bound states, the correct equation is

$$\begin{aligned} T(k',k;k^2) \;=\; & V^\Lambda_{\text{low k}}(k',k) + \int_p T(k',p;p^2) \frac{1}{k^2 - p^2} T^\Lambda(p,k;p^2) \\ & + \sum_i \langle k'|V^\Lambda_{\text{low k}}|\chi^\Lambda_{B_i}\rangle \frac{1}{k^2 + k^2_{B_i}} \langle \tilde{\chi}^\Lambda_{B_i}|V^\Lambda_{\text{low k}}|k\rangle. \end{aligned} \tag{4.7}$$

Similarly, the equation which relate the left-on-shell T-matrix to the right-on-shell one is also modified as

$$\begin{aligned} T^\Lambda(p,k;p^2) \;=\; & T(p,k;k^2) + \int_q T(p,q;q^2) \left\{ \frac{1}{p^2 - q^2} - \frac{1}{k^2 - q^2} \right\} T^\Lambda(q,k;q^2) \\ & + \sum_i \langle p|V^\Lambda_{\text{low k}}|\chi^\Lambda_{B_i}\rangle \left\{ \frac{1}{p^2 + k^2_{B_i}} - \frac{1}{k^2 + k^2_{B_i}} \right\} \langle \tilde{\chi}^\Lambda_{B_i}|V^\Lambda_{\text{low k}}|k\rangle. \end{aligned} \tag{4.8}$$

The extra terms are however cutoff independent because of Eq. (3.21). Thus they do not contribute to the RGE for the $V_{\text{low k}}$.

V. SUMMARY AND DISCUSSIONS

In this paper, we have shown that each of the existent derivations of the RGE, Eq. (1.1), has a serious defect. The derivations based on the "semi-group composition law" are shown to be unjustified, while the derivations based on the completeness must be modified when bound states are present but the resulting RGE is shown to be unmodified. After all, the present work shows that, even though the derivation given in Ref. [14] does not take into account the effects of bound states, Eq. (1.1) is correct irrespective of the existence of bound states, if we assume Eq. (3.20) or, equivalently, Eq. (3.21).

It is important to note that, in order to derive Eq.(1.1), one actually needs to require not only that the half-on-shell T-matrix is preserved, but also that the matrix elements $\langle p|V^{\Lambda}_{\text{low k}}|\chi^{\Lambda}_{B_i}\rangle$ $(0 \leq p < \Lambda)$ is invariant under the change of the cutoff, Λ. This latter requirement does not immediately follow the former. In this sense, the derivation rests on the stronger (and consistent) requirements, Eqs. (3.3) and (3.20), than just requiring the preservation of the half-on-shell T-matrix.

It may be interesting to note that the RGE is not directly derived from the concrete reduction methods such as LS and KLR methods, because the second and the third derivations of Ref. [14] are now shown to be invalid. Still, because LS method preserves the half-on-shell T-matrix (and the matrix elements for the bound states), it should satisfy the same RGE.

It is also interesting to note that the RGE for V_{WRG} is much simpler and more easily derived than that for $V_{\text{low k}}$. Furthermore, since the $V_{\text{low k}}$ RGE is too complicated to get a simple picture about the action under the RG transformations, the usual concepts such as relevant and irrelevant operators do not seem to be useful, while they are actually important in the Wilsonian RG approach [26, 27, 28, 29]. To our best knowledge, no one has ever found the nontrivial fixed point of the RGE for $V_{\text{low k}}$, which plays an essential role for V_{WRG}. We thus suspect that the convergence of "realistic potentials" to a universal one is not a (direct) consequence of "universality" of the RG action.

We conjecture that the convergence is best understood in the light of the inverse scattering problem. (See Chapter 20 of Ref. [32] or Chapter 12 of Ref. [33] for reviews.) Note that all of the "realistic" potentials contain the information about the phase shifts below $E_{lab} \lesssim 350$ MeV, and that the LS reduction preserves the half-on-shell T-matrix elements (hence the phase shifts), as we will show below. Since the reduced theory does not contain high energy

scattering by construction, the information about the phase shifts contained in the reduced potential provides more complete information about the phase shifts "at all energies" as the cutoff is lowered. On the other hand, a result of the inverse problem, in its simplest case, is that the potential is uniquely determined by the phase shifts at all energies. Even though careful analysis is needed to establish the connection between the convergence and the uniqueness of the potential in the inverse problem, it seems to provide a more natural way of understanding the convergence than the RG "universality."

In Ref. [34] it is reported that the conventional way of calculating $V_{\text{low k}}$ shows a slow convergence and a new method based on V_{WRG} is proposed. It seems, however, difficult to transform V_{WRG} to an energy-independent potential analytically, even though there may be no difficulty in doing so numerically. In Ref. [27], such a transformation was considered for a very simple potential as a field redefinition in the path integral formulation. We found that there is a nontrivial Jacobian, which cannot be calculated easily.

Recently, there emerges a new approach [35, 36, 37, 38] based on the Wegner-Glazek-Wilson (WGW) similarity RG transformation [39, 40, 41]. In this formulation, the RGE is given from the outset so that there is no problem with the derivation of the RGE. It is however unclear to us if the WGW similarity transformation preserves the half-on-shell T-matrix. Note that the LS similarity transformation preserves the half-on-shell T-matrix because of the particular property of the "wave operator" ω, $P\omega = 0$, which leads to

$$V_{LS} \equiv P(1-\omega)(H_0 + V_{NN})(1+\omega)P - PH_0P = PV_{NN}(1+\omega)P, \tag{5.1}$$

so that

$$\langle k|V_{LS}|\chi^\Lambda\rangle \;=\; \langle k|PV_{NN}(1+\omega)P|\chi^\Lambda\rangle = \langle k|V_{NN}|\Psi\rangle, \tag{5.2}$$

since $Q|\Psi\rangle = Q\omega P|\Psi\rangle = Q\omega P|\chi^\Lambda\rangle$. On the other hand, WGW similarity transformed potential V_s is defined as

$$V_s \equiv U(s)(T_{\text{rel}} + V)U(s)^\dagger - T_{\text{rel}}, \tag{5.3}$$

where the operator $U(s)$ is chosen in most papers as

$$\frac{dU(s)}{ds}U^\dagger(s) = [T_{\text{rel}}, V_s]. \tag{5.4}$$

One would be interested in the T-matrix of the transformed theory defined as $\langle k|V_s|\chi_s\rangle$, where $|\chi_s\rangle \equiv U(s)|\Psi\rangle$ is the eigenstate of the transformed Hamiltonian. Unlike the LS transformation, the T-matrix does not seem to be invariant under the change of s.

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[1] S. Weinberg, Phys. Lett. **B251**, 288 (1990).

[2] S. Weinberg, Nucl. Phys. **B363**, 3 (1991).

[3] K. G. Wilson and J. B. Kogut, Phys. Rept. **12**, 75 (1974).

[4] S. R. Beane, P. F. Bedaque, W. C. Haxton, D. R. Phillips, and M. J. Savage, in *At the Frontier of Particle Physics, Handbook of QCD*, edited by M. Shifman (World Scientific, 2000), vol. 1, chap. 3, nucl-th/0008064.

[5] P. F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. **52**, 339 (2002), nucl-th/0203055.

[6] E. Epelbaum, Prog. Part. Nucl. Phys. **57**, 654 (2006), nucl-th/0509032.

[7] R. Machleidt (2007), arXiv:0704.0807 [nucl-th].

[8] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. **C49**, 2950 (1994), nucl-th/9406039.

[9] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. **C51**, 38 (1995), nucl-th/9408016.

[10] R. Machleidt, F. Sammarruca, and Y. Song, Phys. Rev. **C53**, 1483 (1996), nucl-th/9510023.

[11] R. Machleidt, Phys. Rev. **C63**, 024001 (2001), nucl-th/0006014.

[12] S. K. Bogner and T. T. S. Kuo, Phys. Lett. **B500**, 279 (2001), nucl-th/0009077.

[13] S. K. Bogner, T. T. S. Kuo, A. Schwenk, D. R. Entem, and R. Machleidt, Phys. Lett. **B576**, 265 (2003), nucl-th/0108041.

[14] S. K. Bogner, A. Schwenk, T. T. S. Kuo, and G. E. Brown (2001), nucl-th/0111042.

[15] S. Bogner, T. T. S. Kuo, L. Coraggio, A. Covello, and N. Itaco, Phys. Rev. **C65**, 051301 (2002).

[16] S. K. Bogner, T. T. S. Kuo, and A. Schwenk, Phys. Rept. **386**, 1 (2003), nucl-th/0305035.

[17] A. Nogga, S. K. Bogner, and A. Schwenk, Phys. Rev. **C70**, 061002 (2004), nucl-th/0405016.

[18] K. A. Brueckner, Phys. Rev. **100**, 36 (1955).

[19] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

[20] H. A. Bethe and J. Goldstone, Proc. R. Soc. London **A238** (1956).

[21] J. Goldstone, Proc. R. Soc. London **A239** (1956).

[22] F. Andreozzi, Phys. Rev. C **54**, 684 (1996).

[23] S. Bogner, D. Furnstahl, and A. Schwenk, private communication.

[24] K. Suzuki and S. Y. Lee, Prog. Theor. Phys. **64**, 2091 (1980).

[25] T. T. S. Kuo, S. Y. Lee, and K. F. Ratcliff, **A176**, 65 (1971).

[26] M. C. Birse, J. A. McGovern, and K. G. Richardson, Phys. Lett. **B464**, 169 (1999), hep-ph/9807302.

[27] K. Harada, K. Inoue, and H. Kubo, Phys. Lett. **B636**, 305 (2006), nucl-th/0511020.

[28] K. Harada and H. Kubo, Nucl. Phys. **B758**, 304 (2006), nucl-th/0605004.

[29] K. Harada, H. Kubo, and A. Ninomiya (2007), nucl-th/0702074.

[30] S. X. Nakamura, Prog. Theor. Phys. **114**, 77 (2005).

[31] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, 1 (1949).

[32] R. G. Newton, *Scattering Thoery of Waves and Particles* (Springer-Verlag, New York, Heidelberg, Berlin, 1982), 2nd ed.

[33] V. De Alfaro and T. Regge, *Potential Scattering* (North-Holland Pub., Amsterdam, 1965).

[34] S. K. Bogner, R. J. Furnstahl, S. Ramanan, and A. Schwenk, Nucl. Phys. **A784**, 79 (2007), nucl-th/0609003.

[35] S. K. Bogner, R. J. Furnstahl, and R. J. Perry, Phys. Rev. **C75**, 061001 (2007), nucl-th/0611045.

[36] S. K. Bogner, R. J. Furnstahl, R. J. Perry, and A. Schwenk, Phys. Lett. **B649**, 488 (2007), nucl-th/0701013.

[37] E. D. Jurgenson, S. K. Bogner, R. J. Furnstahl, and R. J. Perry (2007), arXiv:0711.4252 [nucl-th].

[38] E. Anderson et al. (2008), arXiv:0801.1098 [nucl-th].

[39] F. Wegner, Ann. Phys. (Leipzig) pp. 77–91 (1994).

[40] S. D. Glazek and K. G. Wilson, Phys. Rev. **D48**, 5863 (1993).

[41] S. D. Glazek and K. G. Wilson, Phys. Rev. **D49**, 4214 (1994).

[42] The demonstration below is essentially due to Bogner, Furnstahl, and Schwenk [23].